

The Lambek-Grishin calculus is NP-complete

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Abstract. The Lambek-Grishin calculus **LG** is the symmetric extension of the non-associative Lambek calculus **NL**. In this paper we prove that the derivability problem for **LG** is NP-complete.

1 Introduction

In his 1958 and 1961 papers, Lambek formulated two versions of the *Syntactic Calculus*: in (Lambek, 1958), types are assigned to *strings*, which are then combined by an *associative* operation; in (Lambek, 1961), types are assigned to *phrases* (bracketed strings), and the composition operation is non-associative. We refer to these two versions as **L** and **NL** respectively.

As for generative power, Kandulski (1988) proved that **NL** defines exactly the context-free languages. Pentus (1993) showed that this also holds for associative **L**. As for the complexity of the derivability problem, de Groote (1999) showed that for **NL** this belongs to PTIME; for **L**, Pentus (2003) proves that the problem is NP-complete and Savateev (2009) shows that NP-completeness also holds for the product-free fragment of **L**.

It is well known that some natural language phenomena require generative capacity beyond context-free. Several extensions of the Syntactic Calculus have been proposed to deal with such phenomena. In this paper we look at the Lambek-Grishin calculus **LG** (Moortgat, 2007, 2009). **LG** is a *symmetric* extension of the nonassociative Lambek calculus **NL**. In addition to $\otimes, \backslash, /$ (product, left and right division), **LG** has dual operations \oplus, \oslash, \ominus (coproduct, left and right difference). These two families are related by linear distributivity principles. Melissen (2009) shows that all languages which are the intersection of a context-free language and the permutation closure of a context-free language are recognizable in **LG**. This places the lower bound for **LG** recognition beyond LTAG. The upper bound is still open.

The key result of the present paper is a proof that the derivability problem for **LG** is NP-complete. This will be shown by means of a reduction from SAT.¹

2 Lambek-Grishin calculus

We define the formula language of **LG** as follows.

¹ This paper has been written as a result of my Master thesis supervised by Michael Moortgat. I would like to thank him, Rosalie Iemhoff and Arno Bastenhof for comments and I acknowledge that any errors are my own.

Let Var be a set of *primitive types*, we use lowercase letters to refer to an element of Var . Let *formulas* be constructed using primitive types and the binary connectives $\otimes, /, \backslash, \oplus, \oslash$ and \odot as follows:

$$A, B ::= p \mid A \otimes B \mid A/B \mid B \backslash A \mid A \oplus B \mid A \oslash B \mid B \odot A$$

The sets of *input* and *output structures* are constructed using formulas and the binary structural connectives $\cdot \otimes \cdot, \cdot / \cdot, \cdot \backslash \cdot, \cdot \oplus \cdot, \cdot \oslash \cdot$ and $\cdot \odot \cdot$ as follows:

$$(\text{input}) \quad X, Y ::= A \mid X \cdot \otimes \cdot Y \mid X \cdot \oslash \cdot P \mid P \cdot \odot \cdot X$$

$$(\text{output}) \quad P, Q ::= A \mid P \cdot \oplus \cdot Q \mid P \cdot / \cdot X \mid X \cdot \backslash \cdot P$$

The *sequents* of the calculus are of the form $X \rightarrow P$, and as usual we write $\vdash_{LG} X \rightarrow P$ to indicate that the sequent $X \rightarrow P$ is derivable in **LG**. The axioms and inference rules are presented in Figure 1, where we use the *display logic* from (Goré, 1998), but with different symbols for the *structural connectives*.

It has been proven by Moortgat (2007) that we have *Cut admissibility* for **LG**. This means that for every derivation using the *Cut*-rule, there exists a corresponding derivation that is *Cut-free*. Therefore we will assume that the *Cut*-rule is not needed anywhere in a derivation.

3 Preliminaries

3.1 Derivation length

We will first show that for every derivable sequent there exists a *Cut-free* derivation that is polynomial in the length of the sequent. The length of a sequent φ , denoted as $|\varphi|$, is defined as the number of (formula and structural) connectives used to construct this sequent. A subscript will be used to indicate that we count only certain connectives, for example $|\varphi|_{\otimes}$.

Lemma 1. *If $\vdash_{LG} \varphi$ there exists a derivation with exactly $|\varphi|$ logical rules.*

Proof. If $\vdash_{LG} \varphi$ then there exists a *Cut-free* derivation for φ . Because every logical rule removes one logical connective and there are no rules that introduce logical connectives, this derivation contains $|\varphi|$ logical rules. \square

Lemma 2. *If $\vdash_{LG} \varphi$ there exists a derivation with at most $\frac{1}{4}|\varphi|^2$ Grishin interactions.*

Proof. Let us take a closer look at the Grishin interaction principles. First of all, it is not hard to see that the interactions are irreversible. Also note that the interactions happen between the families of input connectives $\{\otimes, /, \backslash\}$ and output connectives $\{\oplus, \oslash, \odot\}$ and that the Grishin interaction principles are the only rules of inference that apply on both families. So, on any pair of one input and one output connective, at most one Grishin interaction principle can be applied.

$$\frac{\overline{p \rightarrow p} \quad Ax}{\frac{X \rightarrow A \quad A \rightarrow P}{X \rightarrow P} \text{ Cut}}$$

$$\frac{\frac{Y \rightarrow X \cdot \backslash \cdot P}{\frac{X \cdot \otimes \cdot Y \rightarrow P}{X \rightarrow P \cdot / \cdot Y} r} r}{\frac{X \cdot \otimes \cdot Y \rightarrow P}{X \rightarrow P \cdot / \cdot Y} r} \quad \frac{\frac{X \cdot \otimes \cdot Q \rightarrow P}{\frac{X \rightarrow P \cdot \oplus \cdot Q}{P \cdot \otimes \cdot X \rightarrow Q} dr} dr}{\frac{X \cdot \otimes \cdot Q \rightarrow P}{P \cdot \otimes \cdot X \rightarrow Q} dr}$$

(a) Display rules

$$\frac{X \cdot \otimes \cdot Y \rightarrow P \cdot \oplus \cdot Q}{X \cdot \otimes \cdot Q \rightarrow P \cdot / \cdot Y} d \otimes / \quad \frac{X \cdot \otimes \cdot Y \rightarrow P \cdot \oplus \cdot Q}{Y \cdot \otimes \cdot Q \rightarrow X \cdot \backslash \cdot P} d \otimes \backslash$$

$$\frac{X \cdot \otimes \cdot Y \rightarrow P \cdot \oplus \cdot Q}{P \cdot \otimes \cdot X \rightarrow Q \cdot / \cdot Y} d \otimes / \quad \frac{X \cdot \otimes \cdot Y \rightarrow P \cdot \oplus \cdot Q}{P \cdot \otimes \cdot Y \rightarrow X \cdot \backslash \cdot Q} d \otimes \backslash$$

(b) Distributivity rules (Grishin interaction principles)

$$\frac{A \cdot \otimes \cdot B \rightarrow P}{A \otimes B \rightarrow P} \otimes L \quad \frac{X \rightarrow B \cdot \oplus \cdot A}{X \rightarrow B \oplus A} \oplus R$$

$$\frac{X \rightarrow A \cdot / \cdot B}{X \rightarrow A/B} /R \quad \frac{B \cdot \otimes \cdot A \rightarrow P}{B \otimes A \rightarrow P} \otimes L$$

$$\frac{X \rightarrow B \cdot \backslash \cdot A}{X \rightarrow B \backslash A} \backslash R \quad \frac{A \cdot \otimes \cdot B \rightarrow P}{A \otimes B \rightarrow P} \otimes L$$

$$\frac{X \rightarrow A \quad Y \rightarrow B}{X \cdot \otimes \cdot Y \rightarrow A \otimes B} \otimes R \quad \frac{B \rightarrow P \quad A \rightarrow Q}{B \oplus A \rightarrow P \cdot \oplus \cdot Q} \oplus L$$

$$\frac{X \rightarrow A \quad B \rightarrow P}{B/A \rightarrow P \cdot / \cdot X} /L \quad \frac{X \rightarrow B \quad A \rightarrow P}{P \cdot \otimes \cdot X \rightarrow A \otimes B} \otimes R$$

$$\frac{X \rightarrow A \quad B \rightarrow P}{A \backslash B \rightarrow X \cdot \backslash \cdot P} \backslash L \quad \frac{X \rightarrow B \quad A \rightarrow P}{X \cdot \otimes \cdot P \rightarrow B \otimes A} \otimes R$$

(c) Logical rules

Fig. 1: The Lambek-Grishin calculus inference rules

If $\vdash_{LG} \varphi$ there exists a Cut-free derivation of φ . The maximum number of possible Grishin interactions in 1 Cut-free derivation is reached when a Grishin interaction is applied on every pair of one input and one output connective. Thus, the maximum number of Grishin interactions in one Cut-free derivation is $|\varphi|_{\{\otimes, /, \backslash\}} \cdot |\varphi|_{\{\oplus, \odot, \oslash\}}$.

By definition, $|\varphi|_{\{\otimes, /, \backslash\}} + |\varphi|_{\{\oplus, \odot, \oslash\}} = |\varphi|$, so the maximum value of $|\varphi|_{\{\otimes, /, \backslash\}} \cdot |\varphi|_{\{\oplus, \odot, \oslash\}}$ is reached when $|\varphi|_{\{\otimes, /, \backslash\}} = |\varphi|_{\{\oplus, \odot, \oslash\}} = \frac{|\varphi|}{2}$. Then the total number of Grishin interactions in 1 derivation is $\frac{|\varphi|}{2} \cdot \frac{|\varphi|}{2} = \frac{1}{4}|\varphi|^2$, so any Cut-free derivation of φ will contain at most $\frac{1}{4}|\varphi|^2$ Grishin interactions. \square

Lemma 3. *In a derivation of sequent φ at most $2|\varphi|$ display rules are needed to display any of the structural parts.*

Proof. A structural part in sequent φ is nested under at most $|\varphi|$ structural connectives. For each of these connectives, one or two r or dr rules can display the desired part, after which the next connective is visible. Thus, at most $2|\varphi|$ display rules are needed to display any of the structural parts.

Lemma 4. *If $\vdash_{LG} \varphi$ there exists a Cut-free derivation of length $O(|\varphi|^3)$.*

Proof. From Lemma 1 and Lemma 2 we know that there exists a derivation with at most $|\varphi|$ logical rules and $\frac{1}{4}|\varphi|^2$ Grishin interactions. Thus, the derivation consists of $|\varphi| + \frac{1}{4}|\varphi|^2$ rules, with between each pair of consecutive rules the display rules. From Lemma 3 we know that at most $2|\varphi|$ display rules are needed to display any of the structural parts. So, at most $2|\varphi| \cdot (|\varphi| + \frac{1}{4}|\varphi|^2) = 2|\varphi|^2 + \frac{1}{2}|\varphi|^3$ derivation steps are needed in the shortest possible Cut-free derivation for this sequent, and this is in $O(|\varphi|^3)$. \square

3.2 Additional notations

Let us first introduce some additional notations to make the proofs shorter and easier readable.

Let us call an input structure X which does not contain any structural operators except for $\cdot \otimes \cdot$ a \otimes -structure. A \otimes -structure can be seen as a binary tree with $\cdot \otimes \cdot$ in the internal nodes and formulas in the leafs. Formally we define \otimes -structures U and V as:

$$U, V ::= A \mid U \cdot \otimes \cdot V$$

We define $X[]$ and $P[]$ as the input and output structures X and P with a hole in one of their leafs. Formally:

$$X[] ::= [] \mid X[] \cdot \otimes \cdot Y \mid Y \cdot \otimes \cdot X[] \mid X[] \cdot \odot \cdot Q \mid Y \cdot \odot \cdot P[] \mid Q \cdot \odot \cdot X[] \mid P[] \cdot \odot \cdot Y$$

$$P[] ::= [] \mid P[] \cdot \oplus \cdot Q \mid Q \cdot \oplus \cdot P[] \mid P[] \cdot / \cdot Y \mid Q \cdot / \cdot X[] \mid Y \cdot \backslash \cdot P[] \mid X[] \cdot \backslash \cdot Q$$

This notation is similar to the one of de Groote (1999) but with structures. If $X[]$ is a structure with a hole, we write $X[Y]$ for $X[]$ with its hole filled with structure Y . We will write $X^{\otimes}[]$ for a \otimes -structure with a hole.

Furthermore, we extend the definition of hole to formulas, and define $A[]$ as a *formula* A with a hole in it, in a similar manner as for structures. Hence, by $A[B]$ we mean the formula $A[]$ with its hole filled by formula B .

In order to distinguish between input and output polarity formulas, we write A^\bullet for a formula with *input* polarity and A° for a formula with *output* polarity. Note that for structures this is already defined by using X and Y for input polarity and P and Q for output polarity. This can be extended to formulas in a similar way, and we will use this notation only in cases where the polarity is not clear from the context.

3.3 Derived rules of inference

Now we will show and prove some derived rules of inference of **LG**.

Lemma 5. *If $\vdash_{LG} A \rightarrow B$ and we want to derive $X^\otimes[A] \rightarrow P$, we can replace A by B in $X^\otimes[]$. We have the inference rule below:*

$$\frac{A \rightarrow B \quad X^\otimes[B] \rightarrow P}{X^\otimes[A] \rightarrow P} \text{ Repl}$$

Proof. We consider three cases:

1. If $X^\otimes[A] = A$, it is simply the cut-rule:

$$\frac{A \rightarrow B \quad B \rightarrow P}{A \rightarrow P} \text{ Cut}$$

2. If $X^\otimes[A] = Y^\otimes[A] \cdot \otimes \cdot V$, we can move V to the righthand-side and use induction to prove the sequent:

$$\frac{\frac{A \rightarrow B \quad \frac{Y^\otimes[B] \cdot \otimes \cdot V \rightarrow P}{Y^\otimes[B] \rightarrow P \cdot / \cdot V} r}{Y^\otimes[A] \rightarrow P \cdot / \cdot V} \text{ Repl}}{Y^\otimes[A] \cdot \otimes \cdot V \rightarrow P} r$$

3. If $X^\otimes[A] = U \cdot \otimes \cdot Y^\otimes[A]$, we can move U to the righthand-side and use induction to prove the sequent:

$$\frac{\frac{A \rightarrow B \quad \frac{U \cdot \otimes \cdot Y^\otimes[B] \rightarrow P}{Y^\otimes[B] \rightarrow U \cdot \setminus \cdot P} r}{Y^\otimes[A] \rightarrow U \cdot \setminus \cdot P} \text{ Repl}}{U \cdot \otimes \cdot Y^\otimes[A] \rightarrow P} r$$

□

Lemma 6. *If we want to derive $X^\otimes[A \odot B] \rightarrow P$, then we can move the expression $\odot B$ out of the \otimes -structure. We have the inference rule below:*

$$\frac{X^\otimes[A] \cdot \odot \cdot B \rightarrow P}{X^\otimes[A \odot B] \rightarrow P} \text{ Move}$$

Proof. We consider three cases:

1. If $X^\otimes[A \otimes B] = A \otimes B$, then this is simply the $\otimes L$ -rule:

$$\frac{A \cdot \otimes \cdot B \rightarrow Y}{A \otimes B \rightarrow Y} \otimes L$$

2. If $X^\otimes[A \otimes B] = Y^\otimes[A \otimes B] \cdot \otimes \cdot V$, we can move V to the righthand-side and use induction together with the Grishin interaction principles to prove the sequent:

$$\frac{\frac{\frac{(Y^\otimes[A] \cdot \otimes \cdot V) \cdot \otimes \cdot B \rightarrow P}{Y^\otimes[A] \cdot \otimes \cdot V \rightarrow P \cdot \oplus \cdot B} dr}{Y^\otimes[A] \cdot \otimes \cdot B \rightarrow P \cdot / \cdot V} d \otimes /}{\frac{Y^\otimes[A \otimes B] \rightarrow P \cdot / \cdot V}{Y^\otimes[A \otimes B] \cdot \otimes \cdot V \rightarrow P} r} Move$$

3. If $X^\otimes[A \otimes B] = U \cdot \otimes \cdot Y^\otimes[A \otimes B]$, we can move U to the righthand-side and use induction together with the Grishin interaction principles to prove the sequent:

$$\frac{\frac{\frac{(U \cdot \otimes \cdot Y^\otimes[A]) \cdot \otimes \cdot B \rightarrow P}{U \cdot \otimes \cdot Y^\otimes[A] \rightarrow P \cdot \oplus \cdot B} dr}{Y^\otimes[A] \cdot \otimes \cdot B \rightarrow U \cdot \setminus \cdot P} d \otimes \setminus}{\frac{Y^\otimes[A \otimes B] \rightarrow U \cdot \setminus \cdot P}{U \cdot \otimes \cdot Y^\otimes[A \otimes B] \rightarrow P} r} Move$$

□

Lemma 7. $\vdash_{LG} A_1 \otimes (A_2 \otimes \dots (A_{n-1} \otimes A_n)) \rightarrow P \text{ iff } \vdash_{LG} A_1 \cdot \otimes \cdot (A_2 \cdot \otimes \cdot \dots (A_{n-1} \cdot \otimes \cdot A_n)) \rightarrow P$

Proof. The *if*-part can be derived by the application of $n - 1$ times the $\otimes L$ rule together with the r rule:

$$\frac{\frac{\frac{A_1 \cdot \otimes \cdot (A_2 \cdot \otimes \cdot \dots (A_{n-1} \cdot \otimes \cdot A_n)) \rightarrow P}{A_{n-1} \cdot \otimes \cdot A_n \rightarrow \dots \cdot \setminus \cdot (A_2 \cdot \setminus \cdot (A_1 \cdot \setminus \cdot P))} r^*}{A_{n-1} \otimes A_n \rightarrow \dots \cdot \setminus \cdot (A_2 \cdot \setminus \cdot (A_1 \cdot \setminus \cdot P))} \otimes L}{\dots (A_{n-1} \otimes A_n) \rightarrow A_2 \cdot \setminus \cdot (A_1 \cdot \setminus \cdot P)} \dots}{\frac{\dots (A_{n-1} \otimes A_n) \rightarrow A_2 \cdot \setminus \cdot (A_1 \cdot \setminus \cdot P)}{A_2 \cdot \otimes \cdot \dots (A_{n-1} \otimes A_n) \rightarrow A_1 \cdot \setminus \cdot P} r} r}{\frac{A_2 \otimes \dots (A_{n-1} \otimes A_n) \rightarrow A_1 \cdot \setminus \cdot P}{A_1 \cdot \otimes \cdot (A_2 \otimes \dots (A_{n-1} \otimes A_n)) \rightarrow P} r} \otimes L}{A_1 \otimes (A_2 \otimes \dots (A_{n-1} \otimes A_n)) \rightarrow P} \otimes L$$

The *only-if*-part can be derived by application of $n - 1$ times the $\otimes R$ rule followed by a *Cut*:

$$\begin{array}{c}
\frac{A_{n-1} \rightarrow A_{n-1} \quad A_n \rightarrow A_n}{A_{n-1} \cdot \otimes \cdot A_n \rightarrow A_{n-1} \otimes A_n} \otimes R \\
\hline
\frac{A_2 \rightarrow A_2 \quad \dots (A_{n-1} \cdot \otimes \cdot A_n) \rightarrow \dots (A_{n-1} \otimes A_n)}{A_2 \cdot \otimes \dots (A_{n-1} \cdot \otimes \cdot A_n) \rightarrow A_2 \otimes \dots (A_{n-1} \otimes A_n)} \otimes R \quad \dots \\
\hline
\frac{A_1 \rightarrow A_1 \quad A_2 \cdot \otimes \dots (A_{n-1} \cdot \otimes \cdot A_n) \rightarrow A_2 \otimes \dots (A_{n-1} \otimes A_n)}{A_1 \cdot \otimes \cdot (A_2 \cdot \otimes \dots (A_{n-1} \cdot \otimes \cdot A_n)) \rightarrow A_1 \otimes (A_2 \otimes \dots (A_{n-1} \otimes A_n))} \otimes R \quad A_1 \otimes (A_2 \otimes \dots (A_{n-1} \otimes A_n)) \rightarrow P \\
\hline
A_1 \cdot \otimes \cdot (A_2 \cdot \otimes \dots (A_{n-1} \cdot \otimes \cdot A_n)) \rightarrow P \quad Cut
\end{array}$$

Note that because of the Cut elimination theorem, there exists a cut-free derivation for this sequent.

□

3.4 Type similarity

The type similarity relation \sim , introduced by Lambek (1958), is the reflexive transitive symmetric closure of the derivability relation. Formally we define this as:

Definition 1. $A \sim B$ iff there exists a sequence $C_1 \dots C_n$ ($1 \leq i \leq n$) such that $C_1 = A$, $C_n = B$ and $C_i \rightarrow C_{i+1}$ or $C_{i+1} \rightarrow C_i$ for all $1 \leq i < n$.

It was proved by Lambek that $A \sim B$ iff one of the following equivalent statements holds (the so-called *diamond property*):

$$\exists C \text{ such that } A \rightarrow C \text{ and } B \rightarrow C \quad (\text{join})$$

$$\exists D \text{ such that } D \rightarrow A \text{ and } D \rightarrow B \quad (\text{meet})$$

This diamond property will be used in the reduction from SAT to create a choice for a truthvalue of a variable.

Definition 2. If $A \sim B$ and C is the join type of A and B so that $A \rightarrow C$ and $B \rightarrow C$, we define $A \overset{C}{\sqcap} B = (A / ((C/C) \setminus C)) \otimes ((C/C) \setminus B)$ as the meet type of A and B .

This is also the solution given by Lambek (1958) for the associative system **L**, but in fact this is the shortest solution for the non-associative system **NL** (Foret, 2003).

Lemma 8. If $A \sim B$ with join-type C and $\vdash_{LG} A \rightarrow P$ or $\vdash_{LG} B \rightarrow P$, then we also have $\vdash_{LG} A \overset{C}{\sqcap} B \rightarrow P$. We can write this as a derived rule of inference:

$$\frac{A \rightarrow P \quad \text{or} \quad B \rightarrow P}{A \overset{C}{\sqcap} B \rightarrow P} \text{ Meet}$$

Proof.

1. If $A \rightarrow P$:

$$\begin{array}{c}
\frac{C \rightarrow C \quad C \rightarrow C}{C/C \rightarrow C \cdot / \cdot C} /L \\
\frac{C/C \rightarrow C/C}{C/C \rightarrow C/C} /R \quad B \rightarrow C \quad \backslash L \\
\frac{(C/C) \backslash B \rightarrow (C/C) \cdot \backslash \cdot C}{(C/C) \backslash B \rightarrow (C/C) \backslash C} \backslash R \quad A \rightarrow P \\
\frac{A/((C/C) \backslash C) \rightarrow P \cdot / \cdot ((C/C) \backslash B)}{(A/((C/C) \backslash C)) \cdot \otimes \cdot ((C/C) \backslash B) \rightarrow P} /L \quad r \\
\frac{(A/((C/C) \backslash C)) \cdot \otimes \cdot ((C/C) \backslash B) \rightarrow P}{(A/((C/C) \backslash C)) \otimes ((C/C) \backslash B) \rightarrow P} \otimes L
\end{array}$$

2. If $B \rightarrow P$:

$$\begin{array}{c}
\frac{C \rightarrow C \quad C \rightarrow C}{C/C \rightarrow C \cdot / \cdot C} /L \\
\frac{(C/C) \cdot \otimes \cdot C \rightarrow C}{C \rightarrow (C/C) \cdot \backslash \cdot C} r \\
\frac{C \rightarrow (C/C) \cdot \backslash \cdot C}{C \rightarrow (C/C) \backslash C} r \\
A \rightarrow C \quad \frac{C \rightarrow (C/C) \backslash C}{A/((C/C) \backslash C) \rightarrow C \cdot / \cdot C} \backslash R \\
\frac{A/((C/C) \backslash C) \rightarrow C \cdot / \cdot C}{A/((C/C) \backslash C) \rightarrow C/C} /L \\
\frac{A/((C/C) \backslash C) \rightarrow C/C}{(C/C) \backslash B \rightarrow (A/((C/C) \backslash C)) \cdot \backslash \cdot P} /R \quad B \rightarrow P \\
\frac{(C/C) \backslash B \rightarrow (A/((C/C) \backslash C)) \cdot \backslash \cdot P}{(A/((C/C) \backslash C)) \cdot \otimes \cdot ((C/C) \backslash B) \rightarrow P} \backslash L \quad r \\
\frac{(A/((C/C) \backslash C)) \cdot \otimes \cdot ((C/C) \backslash B) \rightarrow P}{(A/((C/C) \backslash C)) \otimes ((C/C) \backslash B) \rightarrow P} \otimes L
\end{array}$$

□

The following lemma is the key lemma of this paper, and its use will become clear to the reader in the construction of Section 4.

Lemma 9. *If $\vdash_{LG} A \overset{C}{\sqcap} B \rightarrow P$ then $\vdash_{LG} A \rightarrow P$ or $\vdash_{LG} B \rightarrow P$, if it is not the case that:*

- $P = P'[A'[(A_1 \otimes A_2)^\circ]]$
- $\vdash_{LG} A/((C/C) \backslash C) \rightarrow A_1$
- $\vdash_{LG} (C/C) \backslash B \rightarrow A_2$

Proof. We have that $\vdash_{LG} (A/((C/C) \backslash C)) \otimes ((C/C) \backslash B) \rightarrow P$, so from Lemma 7 we know that $\vdash_{LG} (A/((C/C) \backslash C)) \cdot \otimes \cdot ((C/C) \backslash B) \rightarrow P$. Remark that this also means that there exists a cut-free derivation for this sequent. By induction on the length of the derivation we will show that if $\vdash_{LG} (A/((C/C) \backslash C)) \cdot \otimes \cdot ((C/C) \backslash B) \rightarrow P$, then $\vdash_{LG} A \rightarrow P$ or $\vdash_{LG} B \rightarrow P$, under the assumption that P is not of the form that is explicitly excluded in this lemma. We will look at the derivations in a top-down way.

The induction base is the case where a logical rule is applied on the lefthand-side of the sequent. At a certain point in the derivation, possibly when P is an atom, one of the following three rules must be applied:

1. The $\otimes R$ rule, but then $P = A_1 \otimes A_2$ and in order to come to a derivation it must be the case that $\vdash_{LG} A/((C/C)\backslash C) \rightarrow A_1$ and $\vdash_{LG} (C/C)\backslash B \rightarrow A_2$. However, this is explicitly excluded in this lemma so this can never be the case.
2. The $/L$ rule, in this case first the r rule is applied so that we have $\vdash_{LG} A/((C/C)\backslash C) \rightarrow P \cdot / \cdot ((C/C)\backslash B)$. Now if the $/L$ rule is applied, we must have that $\vdash_{LG} A \rightarrow P$.
3. The $\backslash L$ rule, in this case first the r rule is applied so that we have $\vdash_{LG} (C/C)\backslash B \rightarrow (A/((C/C)\backslash C)) \cdot \backslash \cdot P$. Now if the $\backslash L$ rule is applied, we must have that $\vdash_{LG} B \rightarrow P$.

The induction step is the case where a logical rule is applied on the righthand-side of the sequent. Let $\delta = \{r, dr, d \otimes /, d \otimes \backslash, d \otimes /, d \otimes \backslash\}$ and let δ^* indicate a (possibly empty) sequence of structural residuation steps and Grishin interactions. For example for the $\otimes R$ rule there are two possibilities:

- The lefthand-side ends up in the first premiss of the $\otimes R$ rule:

$$\frac{\frac{\frac{A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B) \rightarrow P''[A']}{P'[(A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B)] \rightarrow A'} \delta^* \quad B' \rightarrow Q}{\frac{P'[(A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B)] \cdot \otimes \cdot Q \rightarrow A' \otimes B'}{(A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B) \rightarrow P[A' \otimes B']} \otimes R} \delta^*$$

In order to be able to apply the $\otimes R$ rule, we need to have a formula of the form $A' \otimes B'$ on the righthand-side. In the first step all structural rules are applied to display this formula in the righthand-side, and we assume that in the lefthand-side the meet-type ends up in the first structural part (inside a structure with the remaining parts from P that we call P'). After the $\otimes R$ rule has been applied, we can again display our meet-type in the lefthand-side of the formula by moving all other structural parts from P' back to the righthand-side (P'').

In this case it must be that $\vdash_{LG} (A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B) \rightarrow P''[A']$, and by induction we know that in this case also $\vdash_{LG} A \rightarrow P''[A']$ or $\vdash_{LG} B \rightarrow P''[A']$. In the case that $\vdash_{LG} A \rightarrow P''[A']$, we can show that $\vdash_{LG} A \rightarrow P[A' \otimes B']$ as follows:

$$\frac{\frac{A \rightarrow P''[A']}{P'[A] \rightarrow A'} \delta^* \quad B' \rightarrow Q}{\frac{P'[A] \cdot \otimes \cdot Q \rightarrow A' \otimes B'}{A \rightarrow P[A' \otimes B']} \otimes R} \delta^*$$

The case for B is similar.

- The lefthand-side ends up in the second premiss of the $\otimes R$ rule:

$$\frac{\frac{\frac{A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B) \rightarrow P''[B']}{Q \rightarrow A' \quad B' \rightarrow P'[(A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B)]} \delta^*}{\frac{Q \cdot \otimes \cdot P'[(A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B)] \rightarrow A' \otimes B'}{(A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B) \rightarrow P[A' \otimes B']} \otimes R} \delta^*$$

This case is similar to the other case, except that the meet-type ends up in the other premiss. Note that, although in this case it is temporarily moved to the righthand-side, the meet-type will still be in an input polarity position and can therefore be displayed in the lefthand-side again.

In this case it must be that $\vdash_{LG} (A/((C/C)\backslash C)) \cdot \otimes \cdot ((C/C)\backslash B) \rightarrow P''[B']$, and by induction we know that in this case also $\vdash_{LG} A \rightarrow P''[B']$ or $\vdash_{LG} B \rightarrow P''[B']$. In the case that $\vdash_{LG} A \rightarrow P''[B']$, we can show that $\vdash_{LG} A \rightarrow P[A' \odot B']$ as follows:

$$\frac{\frac{Q \rightarrow A' \quad \frac{A \rightarrow P''[B']}{B' \rightarrow P'[A]} \delta^*}{Q \cdot \odot \cdot P'[A] \rightarrow A' \odot B'} \odot R}{A \rightarrow P[A' \odot B']} \delta^*$$

The case for B is similar.

The cases for the other logical rules are similar. □

4 Reduction from SAT to LG

In this section we will show that we can reduce a Boolean formula in conjunctive normal form to a sequent of the *Lambek-Grishin calculus*, so that the corresponding **LG** sequent is provable *if and only if* the CNF formula is satisfiable. This has already been done for the associative system **L** by Pentus (2003) with a similar construction.

Let $\varphi = c_1 \wedge \dots \wedge c_n$ be a Boolean formula in conjunctive normal form with clauses $c_1 \dots c_n$ and variables $x_1 \dots x_m$. For all $1 \leq j \leq m$ let $\neg_0 x_j$ stand for the literal $\neg x_j$ and $\neg_1 x_j$ stand for the literal x_j . Now $\langle t_1, \dots, t_m \rangle \in \{0, 1\}^m$ is a satisfying assignment for φ if and only if for every $1 \leq i \leq n$ there exists a $1 \leq j \leq m$ such that the literal $\neg_{t_j} x_j$ appears in clause c_i .

Let p_i (for $1 \leq i \leq n$) be distinct primitive types from Var . We now define the following families of types:

$$\begin{aligned} E_j^i(t) &\Leftarrow \begin{cases} p_i \odot (p_i \odot p_i) & \text{if } \neg_{t_j} x_j \text{ appears in clause } c_i \\ p_i & \text{otherwise} \end{cases} && \begin{matrix} \text{if } 1 \leq i \leq n, 1 \leq j \leq m \\ \text{and } t \in \{0, 1\} \end{matrix} \\ E_j(t) &\Leftarrow E_j^1(t) \otimes (E_j^2(t) \otimes (\dots (E_j^{n-1}(t) \otimes E_j^n(t)))) && \text{if } 1 \leq j \leq m \text{ and } t \in \{0, 1\} \\ H_j &\Leftarrow p_1 \otimes (p_2 \otimes (\dots (p_{n-1} \otimes p_n))) && \text{if } 1 \leq j \leq m \\ F_j &\Leftarrow E_j(1) \overset{H_j}{\sqcap} E_j(0) && \text{if } 1 \leq j \leq m \\ G_0 &\Leftarrow H_1 \otimes (H_2 \otimes (\dots (H_{m-1} \otimes H_m))) \\ G_i &\Leftarrow G_{i-1} \odot (p_i \odot p_i) && \text{if } 1 \leq i \leq n \end{aligned}$$

Let $\bar{\varphi} = F_1 \otimes (F_2 \otimes (\dots (F_{m-1} \otimes F_m))) \rightarrow G_n$ be the **LG** sequent corresponding to the Boolean formula φ . We now claim that the $\models \varphi$ *if and only if* $\vdash_{LG} \bar{\varphi}$.

4.1 Example

Let us take the Boolean formula $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$ as an example. We have the primitive types $\{p_1, p_2\}$ and the types as shown in Figure 2. The formula is satisfiable (for example with the assignment $\langle 1, 0 \rangle$), thus $\vdash_{LG} F_1 \otimes F_2 \rightarrow G_2$. A sketch of the derivation is given in Figure 2, some parts are proved in lemma's later on.

4.2 Intuition

Let us give some intuitions for the different parts of the construction, and a brief idea of why this would work. The basic idea is that on the lefthand-side we create a type for each literal (F_j is the formula for literal j), which will in the end result in the base type H_j , so $F_1 \otimes (F_2 \otimes (\dots (F_{m-1} \otimes F_m)))$ will result in G_0 . However, on the righthand-side we have an occurrence of the expression $\odot(p_i \odot p_i)$ for each clause i , so in order to come to a derivation, we need to apply the $\odot R$ rule for every clause i .

Each literal on the lefthand-side will result in either $E_j(1)$ (x_j is *true*) or $E_j(0)$ (x_j is *false*). This choice is created using a *join type* H_j such that $\vdash_{LG} E_j(1) \rightarrow H_j$ and $\vdash_{LG} E_j(0) \rightarrow H_j$, which we use to construct the *meet type* F_j . It can be shown that in this case $\vdash_{LG} F_j \rightarrow E_j(1)$ and $\vdash_{LG} F_j \rightarrow E_j(0)$, i.e. in the original formula we can replace F_j by either $E_j(1)$ or $E_j(0)$, giving us a choice for the truthvalue of x_j .

Let us assume that we need $x_1 = \text{true}$ to satisfy the formula, so on the lefthand-side we need to replace F_j by $E_1(1)$. $E_1(1)$ will be the product of exactly n parts, one for each clause ($E_1^1(1) \dots E_1^n(1)$). Here $E_1^i(1)$ is $p_i \odot (p_i \odot p_i)$ iff x_1 does appear in clause i , and p_i otherwise. The first thing that should be noticed is that $\vdash_{LG} p_i \odot (p_i \odot p_i) \rightarrow p_i$, so we can rewrite all $p_i \odot (p_i \odot p_i)$ into p_i so that $\vdash_{LG} E_1(1) \rightarrow H_1$.

However, we can also use the type $p_i \odot (p_i \odot p_i)$ to facilitate the application of the $\odot R$ rule on the occurrence of the expression $\odot(p_i \odot p_i)$ in the righthand-side. From Lemma 6 we know that $\vdash_{LG} X^\otimes[p_i \odot (p_i \odot p_i)] \rightarrow G_i$ if $\vdash_{LG} X^\otimes[p_i] \cdot \odot \cdot (p_i \odot p_i) \rightarrow G_i$, so if the expression $\odot Y$ occurs somewhere in a \otimes -structure we can move it to the outside. Hence, from the occurrence of $p_i \odot (p_i \odot p_i)$ on the lefthand-side we can move $\odot(p_i \odot p_i)$ to the outside of the \otimes -structure and p_i will be left behind within the original structure (just as if we rewrote it to p_i). However, the sequent is now of the form $X^\otimes[p_i] \cdot \odot \cdot (p_i \odot p_i) \rightarrow G_{i-1} \odot (p_i \odot p_i)$, so after applying the $\odot R$ rule we have $X^\otimes[p_i] \rightarrow G_{i-1}$.

Now if the original CNF formula is satisfiable, we can use the meet types on the lefthand-side to derive the correct value of $E_j(1)$ or $E_j(0)$ for all j . If this assignment indeed satisfies the formula, then for each i the formula $p_i \odot (p_i \odot p_i)$ will appear at least once. Hence, for all occurrences of the expression $\odot(p_i \odot p_i)$ on the righthand-side we can apply the $\odot R$ rule, after which the rest of the $p_i \odot (p_i \odot p_i)$ can be rewritten to p_i in order to derive the base type.

If the formula is not satisfiable, then there will be no way to have the $p_i \odot (p_i \odot p_i)$ types on the lefthand-side for *all* i , so there will be at least one occurrence

$$\begin{array}{c}
\frac{\frac{p_1 \rightarrow p_1 \quad p_1 \rightarrow p_1}{p_1 \cdot \odot \cdot p_1 \rightarrow p_1 \odot p_1} \odot R}{p_1 \rightarrow p_1 \cdot \oplus \cdot (p_1 \odot p_1)} dr \\
\frac{p_1 \cdot \odot \cdot (p_1 \odot p_1) \rightarrow p_1}{p_1 \odot (p_1 \odot p_1) \rightarrow p_1} \odot L \\
\frac{p_1 \odot (p_1 \odot p_1) \rightarrow p_1 \quad p_2 \rightarrow p_2}{(p_1 \odot (p_1 \odot p_1)) \cdot \odot \cdot p_2 \rightarrow p_1 \otimes p_2} \otimes R \\
\frac{(p_1 \odot (p_1 \odot p_1)) \cdot \odot \cdot p_2 \rightarrow p_1 \otimes p_2}{(p_1 \odot (p_1 \odot p_1)) \otimes p_2 \rightarrow p_1 \otimes p_2} \otimes L \\
\frac{E_1(1) \rightarrow H_1}{F_1 \rightarrow H_1} 12 \quad \frac{\frac{p_1 \rightarrow p_1 \quad p_2 \rightarrow p_2}{p_1 \cdot \otimes \cdot p_2 \rightarrow p_1 \otimes p_2} \otimes R}{p_1 \cdot \otimes \cdot p_2 \rightarrow H_2} Def \\
\frac{F_1 \cdot \otimes \cdot (p_1 \cdot \otimes \cdot p_2) \rightarrow H_1 \otimes H_2 \quad p_1 \odot p_1 \rightarrow p_1 \odot p_1}{(F_1 \cdot \otimes \cdot (p_1 \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_1 \odot p_1) \rightarrow (H_1 \otimes H_2) \odot (p_1 \odot p_1)} \odot R \\
\frac{(F_1 \cdot \otimes \cdot (p_1 \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_1 \odot p_1) \rightarrow (H_1 \otimes H_2) \odot (p_1 \odot p_1)}{(F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_1 \odot p_1) \rightarrow G_1} Def \\
\frac{F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot p_2) \rightarrow G_1 \quad p_2 \odot p_2 \rightarrow p_2 \odot p_2}{(F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_2 \odot p_2) \rightarrow G_1 \odot (p_2 \odot p_2)} Move \quad \odot R \\
\frac{(F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_2 \odot p_2) \rightarrow G_1 \odot (p_2 \odot p_2)}{(F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_2 \odot p_2) \rightarrow G_2} Def \\
\frac{(F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot p_2)) \cdot \odot \cdot (p_2 \odot p_2) \rightarrow G_2}{F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot (p_2 \odot (p_2 \odot p_2))) \rightarrow G_2} Move \\
\frac{F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot (p_2 \odot (p_2 \odot p_2))) \rightarrow G_2}{(p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot (p_2 \odot (p_2 \odot p_2)) \rightarrow F_1 \cdot \setminus \cdot G_2} r \\
\frac{(p_1 \odot (p_1 \odot p_1)) \cdot \otimes \cdot (p_2 \odot (p_2 \odot p_2)) \rightarrow F_1 \cdot \setminus \cdot G_2}{(p_1 \odot (p_1 \odot p_1)) \otimes (p_2 \odot (p_2 \odot p_2)) \rightarrow F_1 \cdot \setminus \cdot G_2} \otimes L \\
\frac{(p_1 \odot (p_1 \odot p_1)) \otimes (p_2 \odot (p_2 \odot p_2)) \rightarrow F_1 \cdot \setminus \cdot G_2}{F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \otimes (p_2 \odot (p_2 \odot p_2))) \rightarrow G_2} r \\
\frac{F_1 \cdot \otimes \cdot ((p_1 \odot (p_1 \odot p_1)) \otimes (p_2 \odot (p_2 \odot p_2))) \rightarrow G_2}{F_1 \cdot \otimes \cdot E_2(0) \rightarrow G_2} Def \\
\frac{F_1 \cdot \otimes \cdot E_2(0) \rightarrow G_2 \quad \frac{F_2 \rightarrow E_2(0)}{F_2 \rightarrow E_2(0)} 12}{F_1 \otimes F_2 \rightarrow G_2} Repl \\
\frac{F_1 \cdot \otimes \cdot F_2 \rightarrow G_2}{F_1 \otimes F_2 \rightarrow G_2} \otimes L
\end{array}$$

$$\begin{array}{l}
E_1(0) = p_1 \otimes (p_2 \odot (p_2 \odot p_2)) \\
E_1(1) = (p_1 \odot (p_1 \odot p_1)) \otimes p_2 \\
E_2(0) = (p_1 \odot (p_1 \odot p_1)) \otimes (p_2 \odot (p_2 \odot p_2)) \\
E_2(1) = p_1 \otimes p_2 \\
H_1 = p_1 \otimes p_2 \\
H_2 = p_1 \otimes p_2 \\
F_1 = E_1(1) \overset{H_1}{\sqcap} E_1(0) \\
F_2 = E_2(1) \overset{H_2}{\sqcap} E_2(0) \\
G_2 = ((H_1 \otimes H_2) \odot (p_1 \odot p_1)) \odot (p_2 \odot p_2)
\end{array}$$

Fig. 2: Sketch proof for LG sequent corresponding to $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$

of $\odot(p_i \odot p_i)$ on the righthand-side where we cannot apply the $\odot R$ rule. Because the \odot will be the main connective we cannot apply any other rule, and we will never come to a valid derivation.

Note that the meet type F_j provides an *explicit* switch, so we first have to replace it by *either* $E_j(1)$ *or* $E_j(0)$ before we can do anything else with it. This guarantees that if $\vdash_{LG} \bar{\varphi}$, there also must be some assignment $\langle t_1, \dots, t_m \rangle \in \{0, 1\}^m$ such that $\vdash_{LG} E_1(t_1) \otimes (E_2(t_2) \otimes (\dots (E_{m-1}(t_{m-1}) \otimes E_m(t_m)))) \rightarrow G_n$, which means that $\langle t_1, \dots, t_m \rangle$ is a satisfying assignment for φ .

5 Proof

We will now prove the main claim that $\models \varphi$ *if and only if* $\vdash_{LG} \bar{\varphi}$. First we will prove that *if* $\models \varphi$, *then* $\vdash_{LG} \bar{\varphi}$.

5.1 If-part

Let us assume that $\models \varphi$, so there is an assignment $\langle t_1, \dots, t_m \rangle \in \{0, 1\}^m$ that satisfies φ .

Lemma 10. *If $1 \leq i \leq n$, $1 \leq j \leq m$ and $t \in \{0, 1\}$ then $\vdash_{LG} E_j^i(t) \rightarrow p_i$.*

Proof. We consider two cases:

1. If $E_j^i(t) = p_i$ this is simply the axiom rule.
2. If $E_j^i(t) = p_i \odot (p_i \odot p_i)$ we can prove it as follows:

$$\frac{\frac{\frac{p_i \rightarrow p_i \quad p_i \rightarrow p_i}{p_i \cdot \odot \cdot p_i \rightarrow p_i \odot p_i} \odot R}{p_i \rightarrow p_i \cdot \oplus \cdot (p_i \odot p_i)} dr}{\frac{p_i \cdot \odot \cdot (p_i \odot p_i) \rightarrow p_i}{p_i \odot (p_i \odot p_i) \rightarrow p_i} \odot L} dr$$

□

Lemma 11. *If $1 \leq j \leq m$ and $t \in \{0, 1\}$, then $\vdash_{LG} E_j(t) \rightarrow H_j$.*

Proof. From Lemma 7 we know that we can turn $E_j(t)$ into a \otimes -structure. From Lemma 10 we know that $\vdash_{LG} E_j^i(t) \rightarrow p_i$, so using Lemma 5 we can replace all $E_j^i(t)$ by p_i in $E_j(t)$ after which we can apply the $\otimes R$ rule $n - 1$ times to prove the lemma. □

Lemma 12. *If $1 \leq j \leq m$, then $\vdash_{LG} F_j \rightarrow E_j(t_j)$*

Proof. From Lemma 11 we know that $\vdash_{LG} E_j(1) \rightarrow H_j$ and $\vdash_{LG} E_j(0) \rightarrow H_j$, so $E_j(1) \sim E_j(0)$ with join-type H_j . Now from Lemma 8 we know that $\vdash_{LG} E_j(1) \overset{H_j}{\sqcap} E_j(0) \rightarrow E_j(1)$ and $\vdash_{LG} E_j(1) \overset{H_j}{\sqcap} E_j(0) \rightarrow E_j(0)$. □

Lemma 13. We can replace each F_j in $\bar{\varphi}$ by $E_j(t_j)$, so:

$$\frac{E_1(t_1) \cdot \otimes \cdot (E_2(t_2) \cdot \otimes \cdot (\dots (E_{m-1}(t_{m-1}) \cdot \otimes \cdot E_m(t_m)))) \rightarrow G_n}{F_1 \otimes (F_2 \otimes (\dots (F_{m-1} \otimes F_m))) \rightarrow G_n}$$

Proof. This can be proven by using Lemma 7 to turn it into a \otimes -structure, and then apply Lemma 12 in combination with Lemma 5 m times. \square

Lemma 14. In $E_1(t_1) \cdot \otimes \cdot (E_2(t_2) \cdot \otimes \cdot (\dots (E_{m-1}(t_{m-1}) \cdot \otimes \cdot E_m(t_m)))) \rightarrow G_n$, there is at least one occurrence of $p_i \otimes (p_i \otimes p_i)$ in the lefthand-side for every $1 \leq i \leq n$.

Proof. This sequence of $E_1(t_1), \dots, E_m(t_m)$ represents the truthvalue of all variables, and because this is a satisfying assignment, for all i there is at least one index k such that $\neg_{t_k} x_k$ appears in clause i . By definition we have that $E_k^i(t_k) = p_i \otimes (p_i \otimes p_i)$. \square

Definition 3. $Y_j^i \Leftarrow E_j(t_j)$ with every occurrence of $p_k \otimes (p_k \otimes p_k)$ replaced by p_k for all $i < k \leq n$

Lemma 15. $\vdash_{LG} Y_1^0 \cdot \otimes \cdot (Y_2^0 \cdot \otimes \cdot (\dots (Y_{m-1}^0 \cdot \otimes \cdot Y_m^0))) \rightarrow G_0$

Proof. Because $Y_j^0 = H_j$ by definition for all $1 \leq j \leq m$ and $G_0 = H_1 \otimes (H_2 \otimes (\dots (H_{m-1} \otimes H_m)))$, this can be proven by applying the $\otimes R$ rule $m-1$ times. \square

Lemma 16. If $\vdash_{LG} Y_1^{i-1} \cdot \otimes \cdot (Y_2^{i-1} \cdot \otimes \cdot (\dots (Y_{m-1}^{i-1} \cdot \otimes \cdot Y_m^{i-1}))) \rightarrow G_{i-1}$, then $\vdash_{LG} Y_1^i \cdot \otimes \cdot (Y_2^i \cdot \otimes \cdot (\dots (Y_{m-1}^i \cdot \otimes \cdot Y_m^i))) \rightarrow G_i$

Proof. From Lemma 14 we know that $p_i \otimes (p_i \otimes p_i)$ occurs in $Y_1^i \cdot \otimes \cdot (Y_2^i \cdot \otimes \cdot (\dots (Y_{m-1}^i \cdot \otimes \cdot Y_m^i)))$ (because the Y_j^i parts are $E_j(t_j)$ but with $p_k \otimes (p_k \otimes p_k)$ replaced by p_k only for $k > i$). Using Lemma 6 we can move the expression $\otimes(p_i \otimes p_i)$ to the outside of the lefthand-side of the sequent, after which we can apply the $\otimes R$ -rule. After this we can replace all other occurrences of $p_i \otimes (p_i \otimes p_i)$ by p_i using Lemma 10 and Lemma 5. This process can be summarized as:

$$\frac{\frac{Y_1^{i-1} \cdot \otimes \cdot (Y_2^{i-1} \cdot \otimes \cdot (\dots (Y_{m-1}^{i-1} \cdot \otimes \cdot Y_m^{i-1}))) \rightarrow G_{i-1} \quad p_i \otimes p_i \rightarrow p_i \otimes p_i}{(Y_1^{i-1} \cdot \otimes \cdot (Y_2^{i-1} \cdot \otimes \cdot (\dots (Y_{m-1}^{i-1} \cdot \otimes \cdot Y_m^{i-1}))) \cdot \otimes \cdot (p_i \otimes p_i) \rightarrow G_{i-1} \otimes (p_i \otimes p_i))} \otimes R}{\frac{Y_1^{i-1} \cdot \otimes \cdot (Y_2^{i-1} \cdot \otimes \cdot (\dots (Y_{m-1}^{i-1} \cdot \otimes \cdot Y_m^{i-1}))) \cdot \otimes \cdot (p_i \otimes p_i) \rightarrow G_i}{Y_1^i \cdot \otimes \cdot (Y_2^i \cdot \otimes \cdot (\dots (Y_{m-1}^i \cdot \otimes \cdot Y_m^i))) \rightarrow G_i} 14, 6, 10, 5} Def$$

\square

Lemma 17. $\vdash_{LG} Y_1^n \cdot \otimes \cdot (Y_2^n \cdot \otimes \cdot (\dots (Y_{m-1}^n \cdot \otimes \cdot Y_m^n))) \rightarrow G_n$

Proof. We can prove this using induction with Lemma 15 as base and Lemma 16 as induction step. \square

Lemma 18. If $\models \varphi$, then $\vdash_{LG} \bar{\varphi}$,

Proof. From Lemma 17 we know that $\vdash_{LG} Y_1^n \cdot \otimes \cdot (Y_2^n \cdot \otimes \cdot (\dots (Y_{m-1}^n \cdot \otimes \cdot Y_m^n))) \rightarrow G_n$, and because by definition $Y_j^n = E_j(t_j)$, we also have that $\vdash_{LG} E_1(t_1) \cdot \otimes \cdot (E_2(t_2) \cdot \otimes \cdot (\dots (E_{m-1}(t_{m-1}) \cdot \otimes \cdot E_m(t_m)))) \rightarrow G_n$. Finally combining this with Lemma 13 we have that $\vdash_{LG} \bar{\varphi} = F_1 \otimes (F_2 \otimes (\dots (F_{m-1} \otimes F_m))) \rightarrow G_n$, using the assumption that $\models \varphi$. \square

5.2 Only-if part

For the only if part we will need to prove that *if $\vdash_{LG} \bar{\varphi}$, then $\models \varphi$* . Let us now assume that $\vdash_{LG} \bar{\varphi}$.

Lemma 19. *If $\vdash_{LG} X \rightarrow P'[(P \otimes Y)^\circ]$, then there exist a Q such that Q is part of X or P' (possibly inside a formula in X or P') and $\vdash_{LG} Y \rightarrow Q$.*

Proof. The only rule that matches a \otimes in the righthand-side is the $\otimes R$ rule, so somewhere in the derivation this rule must be applied on the occurrence of $P \otimes Y$. Because this rule needs a $\cdot \otimes \cdot$ connective in the lefthand-side, we know that if $\vdash_{LG} X \rightarrow P'[(P \otimes Y)^\circ]$ it must be the case that we can turn this into $X' \cdot \otimes \cdot Q \rightarrow P \otimes Y$ such that $\vdash_{LG} Y \rightarrow Q$. \square

Lemma 20. *If $\vdash_{LG} E_1(t_1) \cdot \otimes \cdot (E_2(t_2) \cdot \otimes \cdot (\dots (E_{m-1}(t_{m-1}) \cdot \otimes \cdot E_m(t_m)))) \rightarrow G_n$, then there is an occurrence $p_i \otimes (p_i \otimes p_i)$ on the lefthand-side at least once for all $1 \leq i \leq n$.*

Proof. G_n by definition contains an occurrence of the expression $\otimes(p_i \otimes p_i)$ for all $1 \leq i \leq n$. From Lemma 19 we know that somewhere in the sequent we need an occurrence of a structure Q such that $\vdash_{LG} p_i \otimes p_i \rightarrow Q$. From the construction it is obvious that the only possible type for Q is in this case $p_i \otimes p_i$, and it came from the occurrence of $p_i \otimes (p_i \otimes p_i)$ on the lefthand-side. \square

Lemma 21. *If $\vdash_{LG} E_1(t_1) \cdot \otimes \cdot (E_2(t_2) \cdot \otimes \cdot (\dots (E_{m-1}(t_{m-1}) \cdot \otimes \cdot E_m(t_m)))) \rightarrow G_n$, then $\langle t_1, t_2, \dots, t_{m-1}, t_m \rangle$ is a satisfying assignment for the CNF formula.*

Proof. From Lemma 20 we know that there is a $p_i \otimes (p_i \otimes p_i)$ in the lefthand-side of the formula for all $1 \leq i \leq n$. From the definition we know that for each i there is an index j such that $E_j^i(t_j) = p_i \otimes (p_i \otimes p_i)$, and this means that $\neg_{t_j} x_j$ appears in clause i , so all clauses are satisfied. Hence, this choice of $t_1 \dots t_m$ is a satisfying assignment. \square

Lemma 22. *If $1 \leq j \leq m$ and $\vdash_{LG} X^\otimes[F_j] \rightarrow G_n$, then $\vdash_{LG} X^\otimes[E_j(0)] \rightarrow G_n$ or $\vdash_{LG} X^\otimes[E_j(1)] \rightarrow G_n$.*

Proof. We know that $X^\otimes[F_j]$ is a \otimes -structure, so we can apply the r rule several times to move all but the F_j -part to the righthand-side. We then have that $\vdash_{LG} F_j \rightarrow \dots \cdot \setminus \cdot G_n \cdot / \cdot \dots$. From Lemma 9 we know that we now have that $\vdash_{LG} E_j(0) \rightarrow \dots \cdot \setminus \cdot G_n \cdot / \cdot \dots$ or $\vdash_{LG} E_j(1) \rightarrow \dots \cdot \setminus \cdot G_n \cdot / \cdot \dots$. Finally we can apply the r rule again to move all parts back to the lefthand-side, to show that $\vdash_{LG} X^\otimes[E_j(0)] \rightarrow G_n$ or $\vdash_{LG} X^\otimes[E_j(1)] \rightarrow G_n$.

Note that, in order for Lemma 9 to apply, we have to show that this sequent satisfies the constraints. G_n does contain $A_1 \otimes A_2$ with output polarity, however the only connectives in A_1 and A_2 are \otimes . Because no rules apply on $A/((C/C)\backslash C) \rightarrow A'_1 \otimes A''_1$, we have that $\nvdash_{LG} A/((C/C)\backslash C) \rightarrow A_1$. In $X^\otimes[]$, the only \otimes connectives are within other F_k , however these have an input polarity and do not break the constraints either.

So, in all cases F_j provides an *explicit switch*, which means that the truthvalue of a variable can only be changed in all clauses simultaneously. \square

Lemma 23. *If $\vdash_{LG} \bar{\varphi}$, then $\models \varphi$.*

Proof. From Lemma 22 we know that all derivations will first need to replace each F_j by *either* $E_j(1)$ *or* $E_j(0)$. This means that if $\vdash_{LG} F_1 \otimes (F_2 \otimes (\dots (F_{m-1} \otimes F_m))) \rightarrow G_n$, then also $\vdash_{LG} E_1(t_1) \cdot \otimes \cdot (E_2(t_2) \cdot \otimes \cdot (\dots (E_{m-1}(t_{m-1}) \cdot \otimes \cdot E_m(t_m))) \rightarrow G_n$ for some $\langle t_1, t_2, \dots, t_{m-1}, t_m \rangle \in \{0, 1\}^m$. From Lemma 21 we know that this is a satisfying assignment for φ , so if we assume that $\vdash_{LG} \bar{\varphi}$, then $\models \varphi$. \square

5.3 Conclusion

Theorem 1. *LG is NP-complete.*

Proof. From Lemma 4 we know that for every derivable sequent there exists a proof that is of polynomial length, so the derivability problem for **LG** is in *NP*. From Lemma 18 and Lemma 23 we can conclude that we can reduce SAT to **LG**. Because SAT is a known NP-hard problem (Garey and Johnson, 1979), and our reduction is polynomial, we can conclude that derivability for **LG** is also NP-hard.

Combining these two facts we conclude that the derivability problem for **LG** is NP-complete. \square

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